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On annulus twists

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ABSTRACT. We survey some results about annulus twists which are related to Dehn surgery on knots, knot concordance, and 4-manifold theory.

1. Introduction

Lickorish [15] and Wallace [22] proved that every closed connected orientable 3-manifold can be obtained by Dehn surgery on some link in S^3. In other words, every closed connected orientable 3-manifold is described by a framed link in S^3. Our interest is in uniqueness of framed link descriptions of a given 3-manifold. A natural question is the following.

Question 1. If two framed links $\mathcal{L}$ and $\mathcal{L}'$ give the same 3-manifold, then are $\mathcal{L}$ and $\mathcal{L}'$ isotopic as framed links ?

It is well-known that the answer of Question 1 is NO. Indeed, for a given framed link $\mathcal{L}$ and a $1/n$-framed unknot $\mathcal{O}$, two framed links $\mathcal{L}$ and $\mathcal{L} \sqcup \mathcal{O}$ give the same 3-manifold. A modified question is the following.

Question 2. If two framed knots $\mathcal{K}$ and $\mathcal{K}'$ give the same 3-manifold, then are $\mathcal{K}$ and $\mathcal{K}'$ isotopic as framed knots ?

The answer of Question 2 is again NO [16] (see also [8, 9, 17, 20]). The remaining questions are the following.

(1) Under what conditions, are framed knot descriptions of a 3-manifold unique?
(2) To what extent, are framed knot descriptions of a 3-manifold far from unique ?

For the question (1), for example, see [12, 13, 14, 18]. We concentrate on the question (2). More precisely, we consider Clark's problem in Kirby problem list [10]:

Problem 3.6(D). Fix an integer n. Is there a homology 3-sphere (or any 3-manifold) which can be obtained by n-surgery on an infinite number of distinct knots?

In [19], Osoinach solved Problem 3.6(D) for the case $n = 0$ by constructing knots using the method of twisting along an annulus, which we call an annulus twist. After Teragaito's work [21] (see also [7, 11]), Jong, Luecke, Osoinach, and the author [3] solved Problem 3.6(D) affirmatively, where they generalized annulus twists.

In [4], a 4-dimensional extension of Problem 3.6(D) was proposed as follows:

Problem 1. Let n be an integer. Find infinitely many mutually distinct knots K_1, K_2, $\cdots$ such that $X_{K_i}(n) \approx X_{K_j}(n)$ for each $i, j \in \mathbb{N}$.

Here $X_K(n)$ denotes the smooth 4-manifold obtained from the 4-ball B^4 by attaching a 2-handle along K with framing n, and the symbol $\approx$ stands for a diffeomorphism. Due to Akbulut [5, 6], there exists a pair of distinct knots K_n and K'_n such that $X_{K_n}(n) \approx X_{K'_n}(n)$ for each $n \in \mathbb{Z}$, which is a partial answer to Problem 1. In [4], Jong, Omae, Takeuch, and the author solved Problem 1 for the case $n = 0, \pm 4$. In [3], Jong, Luecke, Osoinach, and the author also solved Problem 1 affirmatively.

2. Osoinach's result

In this section, we recall Osoinach's result in [19]. Let K_n be the knots in Figure 1, which is isotopic to the knots in the page 731 in [19]. One of the main results in [19] is the following.

Theorem 2.1 (Osoinach [19]). *We have the following.*
(1) The 3-manifold obtained by 0*-surgery of* K_0 *is toroidal.*
(2) The sequence $\{K_n\}$ *contains infinitely many distinct hyperbolic knots.*
(3) $S^3_0(K_0) \approx S^3_0(K_1) \approx S^3_0(K_2) \approx S^3_0(K_3) \approx \cdots$*, where* $S^3_n(K)$ *denotes the 3-manifold obtained by* n*-surgery of a knot* K *in* S^3*.*

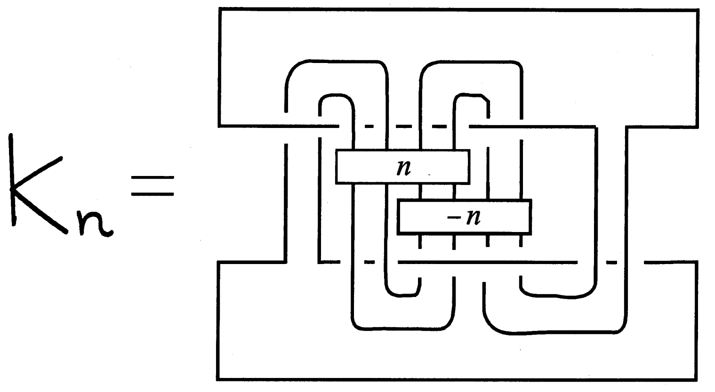

FIGURE 1. The definition of the knots K_n.

Note that we can check that K_n and K_{-n} are isotopic, and Takioka proved that the knots K_n $(n \geq 0)$ are mutually distinct by calculating the Gamma polynomial which is a specialization of the HOMFLYPT polynomial.

Let V be the solid torus standardly embedded in S^3 and V' the 3-manifold as in Figure 2. The main observation in [19] is the following.

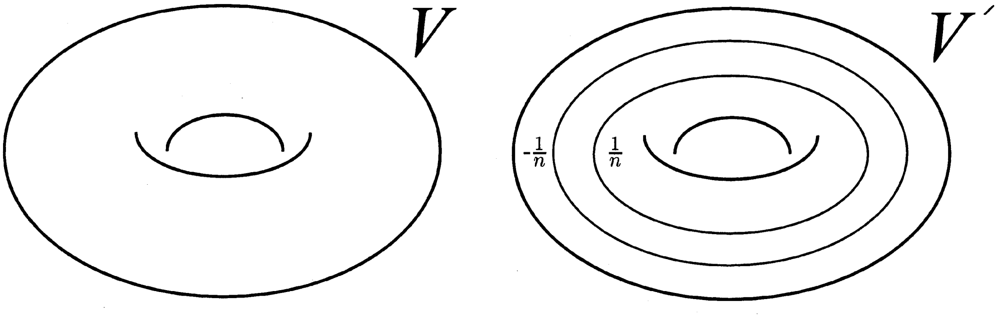

FIGURE 2. The definitions of V and V'.

Lemma 2.2 (cf. Theorem 2.1 in [19]). *There exists a (natural) diffeomorphism*

$$\varphi_n : V' \longrightarrow V$$

such that $\varphi_n|_{\partial V'} = id$.

Remark 2.3. *Osoinach* [19] *considered the diffeomorphism* φ_n^{-1}.

Figure 2 explains a proof of (3) in Theorem 2.1. Note that, by Lemma 2.2, the picture on the bottom-left is diffeomorphic to $S^3_0(K_0)$.

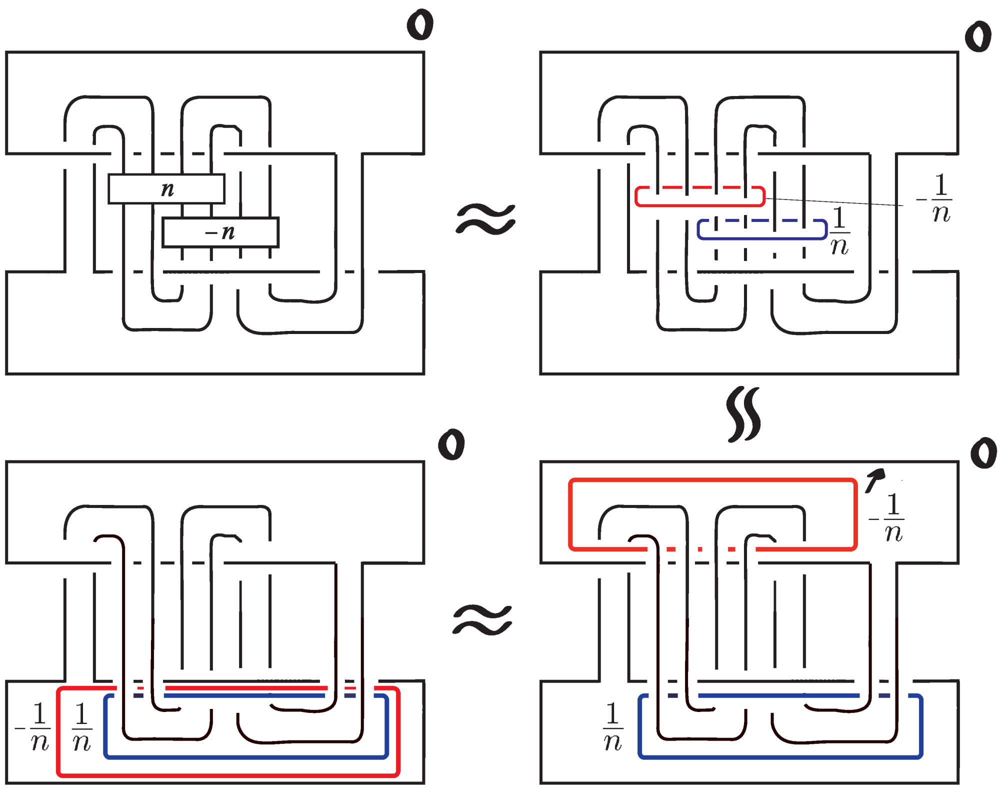

FIGURE 3. A proof of (3) in Theorem 2.1.

3. DEHN SURGERY AND KNOT CONCORDANCE

We recall a terminology in knot concordance. Two knots K and K' are *concordant* if they cobound a properly embedded annulus in $S^3 \times I$. In this paper, we do NOT consider orientations of a given knot.

Dehn surgery on knots and knot concordance are closely related. A motivating question is the following.

Question 3.1 (A.Levine [24]). *If K is concordant to K', then for all n, $S^3_n(K)$ is homology cobordant to $S^3_n(K')$. Is the converse true?*

The following conjecture is due to Akbulut and Kirby (see Problem 1.19 in the Kirby's problem list [10]).

Conjecture. If 0-framed surgeries on two knots give the same 3-manifold, then the knots are concordant.

Tagami and the author [2] proved that Akbulut-Kirby's conjecture is false if the slice-ribbon conjecture is true. Subsequently, Yasui [23] proved that Akbulut-Kirby's conjecture is false by constructing knots K and K' satisfying

(1) $X_K(0)$ and $X_{K'}(0)$ are exotic (i.e. homeomorphic but non-diffeomorphic).
(2) K and K' are not concordant.

Note that $X_K(0)$ and $X_{K'}(0)$ are related by a cork twist. For the details, see [23]. The remaining conjecture is the following.

Conjecture. Let K and K' be knots. If $X_K(0)$ and $X_{K'}(0)$ are diffeomorphic, then K and K' are concordant.

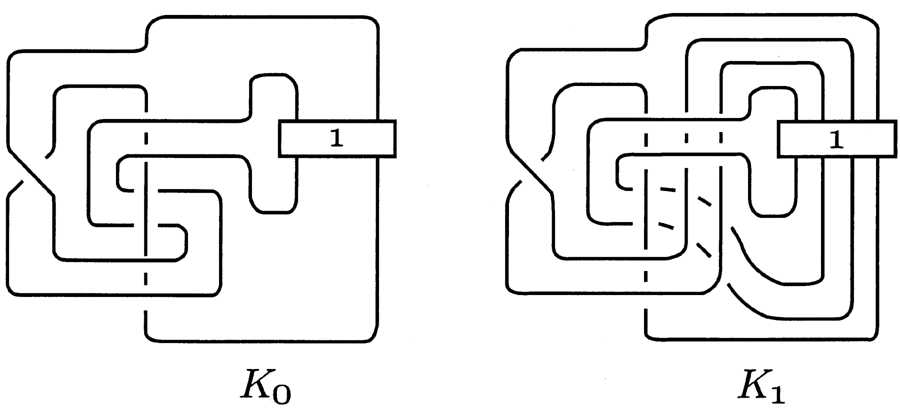

FIGURE 4. The definition of K_0 and K_1.

Remark: Let K_0 and K_1 be the knots in Figure 4. By the result in [4], the 4-manifolds $X_{K_0}(0)$ and $X_{K_1}(0)$ are diffeomorphic. Furthermore, if the slice-ribbon conjecture is true, K_0 and K_1 are not concordant (see [2]).

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